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Monograph

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Entire functions of several variables of bounded index and PDE's

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## List of notation

Let  $\mathbb{R}_+ := (0; +\infty)$ ,  $F$  be an entire function in  $\mathbb{C}^n$ ,  $L : \mathbb{C}^n \rightarrow \mathbb{R}_+$  be a continuous function,  $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ ,  $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$  be a given direction,  $\mathbf{K} = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$ ,  $R = (r_1, \dots, r_n) \in \mathbb{R}_+^n$ .

- $\mathbf{K}! = k_1!k_2! \cdots k_n!$
- $\bar{\mathbf{b}} = (\bar{b}_1, \bar{b}_2, \dots, \bar{b}_n)$  be a conjugate vector to  $\mathbf{b} \in \mathbb{C}^n$ .
- $\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{j=1}^n a_j \bar{b}_j$ ,  $\mathbf{a}^{\mathbf{b}} = a_1^{b_1} a_2^{b_2} \dots a_n^{b_n}$  for  $\mathbf{a}, \mathbf{b} \in \mathbb{C}^n$ .
- $|\mathbf{z}| = \sqrt{|z_1|^2 + |z_2|^2 + \dots + |z_n|^2}$  be the euclidean norm of  $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$ .
- $\|\mathbf{K}\| = k_1 + \dots + k_n$  for  $\mathbf{K} = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$ .
- $D^n(z^0, R) = \{z \in \mathbb{C}^n : |z_j - z_j^0| < r_j, j = 1, \dots, n\}$  be an open polydisc.
- $D^n[z^0, R] = \{z \in \mathbb{C}^n : |z_j - z_j^0| \leq r_j, j = 1, \dots, n\}$  be a closed polydisc.
- $T^n(z^0, R) = \{z \in \mathbb{C}^n : |z_j - z_j^0| = r_j, j = 1, \dots, n\}$  be the skeleton of the polydisc.
- $\frac{\partial^{\|\mathbf{K}\|} F}{\partial Z^{\mathbf{K}}} = \frac{\partial^{k_1 + \dots + k_n} F}{\partial z_1^{k_1} \dots \partial z_n^{k_n}}$  be a partial derivative.
- $\mathbf{0} = (0, \dots, 0)$  be the zero vector.
- $\mathbf{e} = (1, \dots, 1)$ ,  $\mathbf{e}_j = (0, \dots, 0, \underbrace{1}_{j\text{-th place}}, 0, \dots, 0)$ .
- for  $\eta \geq 0$ ,  $z \in \mathbb{C}^n$ ,  $t_0 \in \mathbb{C}$  we define

$$\lambda_1^{\mathbf{b}}(z, t_0, \eta) = \inf \left\{ \frac{L(z + t\mathbf{b})}{L(z + t_0\mathbf{b})} : |t - t_0| \leq \frac{\eta}{L(z + t_0\mathbf{b})} \right\},$$

$$\lambda_2^{\mathbf{b}}(z, t_0, \eta) = \sup \left\{ \frac{L(z + t\mathbf{b})}{L(z + t_0\mathbf{b})} : |t - t_0| \leq \frac{\eta}{L(z + t_0\mathbf{b})} \right\}.$$

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- $\lambda_1^{\mathbf{b}}(z, \eta) = \inf\{\lambda_1^{\mathbf{b}}(z, t_0, \eta) : t_0 \in \mathbb{C}\}$ ,  $\lambda_1^{\mathbf{b}}(\eta) = \inf\{\lambda_1^{\mathbf{b}}(z, \eta) : z \in \mathbb{C}^n\}$ .
  - $\lambda_2^{\mathbf{b}}(z, \eta) = \sup\{\lambda_2^{\mathbf{b}}(z, t_0, \eta) : t_0 \in \mathbb{C}\}$ ,  $\lambda_2^{\mathbf{b}}(\eta) = \sup\{\lambda_2^{\mathbf{b}}(z, \eta) : z \in \mathbb{C}^n\}$ .
  - $Q_{\mathbf{b}}^n$  be a class of functions  $L$ , which for all  $\eta \geq 0$  satisfy a condition  $0 < \lambda_1^{\mathbf{b}}(\eta) \leq \lambda_2^{\mathbf{b}}(\eta) < +\infty$ ;  $Q \equiv Q_1^1$ .
  - For a given  $z^0 \in \mathbb{C}^n$  we denote  $g_{z^0}(t) := F(z^0 + t\mathbf{b})$ . If  $g_{z^0}(t) \neq 0$  for all  $t \in \mathbb{C}$ , then  $G_r^{\mathbf{b}}(F, z^0) := \emptyset$ ; if  $g_{z^0}(t) \equiv 0$ , then  $G_r^{\mathbf{b}}(F, z^0) := \{z^0 + t\mathbf{b} : t \in \mathbb{C}\}$ , and if  $g_{z^0}(t) \not\equiv 0$  and  $a_k^0$  are zeros of  $g_{z^0}(t)$ , then

$$G_r^{\mathbf{b}}(F, z^0) := \bigcup_k \left\{ z^0 + t\mathbf{b} : |t - a_k^0| \leq \frac{r}{L(z^0 + a_k^0 \mathbf{b})} \right\}, \quad r > 0.$$

- $G_r^{\mathbf{b}}(F) := \bigcup_{z^0 \in \mathbb{C}^n} G_r^{\mathbf{b}}(F, z^0)$ .
- $n(r, z^0, t_0, 1/F) = \sum_{|a_k^0 - t_0| \leq r} 1$  be a counting function of the zero sequence  $(a_k^0)$  for  $F(z^0 + t\mathbf{b}) \not\equiv 0$ .
- $M(r, F, z) = \max\{|F(z + t\mathbf{b})| : |t| = r\}$ , where  $t \in \mathbb{C}$ ,  $z \in \mathbb{C}^n$ .
- $L \asymp L^*$  means that for some  $\theta_1, \theta_2 \in \mathbb{R}_+$ ,  $0 < \theta_1 \leq \theta_2 < +\infty$  and for all  $z \in \mathbb{C}^n$  the inequalities  $\theta_1 L(z) \leq L^*(z) \leq \theta_2 L(z)$  hold.
- $g_z(t) = F(z + t\mathbf{b})$  and  $l_z(t) = L(z + t\mathbf{b})$ , where  $z \in \mathbb{C}^n$ ,  $t \in \mathbb{C}$ .
- $\frac{\partial F(z)}{\partial \mathbf{b}} = \sum_{j=1}^n \frac{\partial F(z)}{\partial z_j} b_j = \langle \mathbf{grad} F, \overline{\mathbf{b}} \rangle$ ,  $\frac{\partial^k F(z)}{\partial \mathbf{b}^k} = \frac{\partial}{\partial \mathbf{b}} \left( \frac{\partial^{k-1} F(z)}{\partial \mathbf{b}^{k-1}} \right)$ ,  $k \geq 2$ .
- $T_m(z, \tau) = T_m(z, \tau, F, L, \mathbf{b}) = \frac{1}{m! L^m(\tau)} \left| \frac{\partial^m F(z)}{\partial \mathbf{b}^m} \right|$ ,  $T_m(z) = T_m(z, z)$ .
- $BLID_{\mathbf{b}} \equiv$  “a function of bounded  $L$ -index in the direction  $\mathbf{b}$ ”.
- $BII \equiv$  ”a function of bounded  $l$ -index“.
- $BLIJ \equiv$  “a function of bounded  $\mathbf{L}$ -index in joint variables”.

In addition to the above we introduce some additional notations in the following sections.

## Introduction

In the modern theory of functions of several complex variables, a leading role is played by the theory of entire functions. Methods of investigation of entire functions of several complex variables can be divided into several groups. One of them is based on those properties which can be obtained from the properties of entire functions of one variable, considering this entire function  $F$  as entire function in each variable separately. Other methods are arisen in the study of so-called slice function i.e. entire functions of one variable  $g(\tau) = F(a + b\tau)$ ,  $\tau \in \mathbb{C}$ , which is the restriction of the entire function  $F$  to an arbitrary complex line  $\{z = a + b\tau : \tau \in \mathbb{C}\}$ ,  $a, b \in \mathbb{C}^n$ . This approach is fundamental in our monograph.

B. Lepson ([? ]) investigated properties of entire solutions of linear differential equations and introduced a new subclass of entire functions so-called functions of bounded index. This term is used for the entire functions  $f$  for which there exists  $N \in \mathbb{Z}_+$  such that for all  $p \in \mathbb{Z}_+$  and all  $z \in \mathbb{C}$

$$\frac{|f^{(p)}(z)|}{p!} \leq \max \left\{ \frac{|f^{(k)}(z)|}{k!} : 0 \leq k \leq N \right\}.$$

These functions have been used in the theory value distribution and differential equations (see bibliography in [? ]). In particular, every entire function is a function of bounded value distribution if and only if its derivative is a function of bounded index ([? ]), and every entire solution of the differential equation  $f^{(n)}(t) + \sum_{j=0}^{n-1} a_j f^{(j)}(t) = 0$  is a function of bounded index ([? ]).

G. Fricke and S. Shah investigated an index boundedness of entire solutions of differential equations ([? ]). Later S. Shah ([? ]) and W. Hayman ([? ]) independently proved that every entire function of bounded index is a function of exponential type that its growth is not higher than of normal type

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of first order. M. Salmassi generalized this concept for entire functions of two variables ([? ?]).

To go beyond the class of entire functions of exponential type A. D. Kuzyk and M. M. Sheremeta ([? ], see also [? ]) for a continuous function  $l : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  introduced a concept of entire functions of bounded  $l$ -index, replacing in the previous definition the quantity  $\frac{|f^{(j)}(z)|}{j!}$  by  $\frac{|f^{(j)}(z)|}{j!l^j(|z|)}$ .

The multidimensional case is more difficult so there is no such extensive bibliography, as in one-dimensional. Definition of an entire function of bounded index in several variables was proposed by H. Krishna and S. Shah in their paper ([? ]).

Properties of bivariate functions of bounded index were studied in the paper of M. Salmassi ([? ]). A concept of the entire function of bounded **L**-index in joint variables was introduced by M. M. Sheremeta and M. T. Bordulyak ([? ]). These authors (G. Krishna, S. Shah, M. Salmassi, M. Bordulyak, M. Sheremeta) implemented the first approach to transfer the concept of an entire function of bounded index and of bounded  $l$ -index of one variable to the class of entire functions of several variables. In this case instead of derivatives in the definition, the partial derivatives are considered.

In this way, there was proved a number of analogues of theorems that describe properties of entire functions of bounded **L**-index and criteria of boundedness of **L**-index for entire functions of several variables. And there were obtained sufficient conditions of **L**-index boundedness of entire solutions of some systems of linear differential equations ([? ]). But this approach does not allow to obtain analogues of the one-dimensional characterization of function of bounded **L**-index in terms of behaviour the logarithmic derivative outside zero sets. In particular, attempts to investigate of **L**-index boundedness for some important classes of entire functions (for example infinite products with "planar" zeros) were unsuccessful by technical difficulties.

On the other hand, this approach fits to study, for example, entire functions of the form  $F(z) = f_1(z_1)f_2(z_2)\cdots f_n(z_n)$ ,

$F(z) = f(z_1 + z_2 + \cdots + z_n)$  etc.

Accordingly, the problem arises to consider and to explore an entire function in several variables of bounded  $L$ -index using a second approach.

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